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Study of Some Type of Matrix Fractional Integral

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Abstract: **In this paper, based on a new multiplication of fractional analytic functions, we find some type of matrix fractional integral by using some methods. In fact, our result is a generalization of traditional calculus result.**

Keywords: **New multiplication, fractional analytic functions, matrix fractional integral.**

I. INTRODUCTION

In recent decades, the applications of fractional calculus in various fields of science is growing rapidly, such as physics, biology, mechanics, electrical engineering, viscoelasticity, control theory, modelling, economics, etc [1-15]. However, the rule of fractional derivative is not unique, many scholars have given the definitions of fractional derivatives. The common definition is Riemann-Liouville (R-L) fractional derivative. Other useful definitions include Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative, and Jumarie type of R-L fractional derivative to avoid non-zero fractional derivative of constant function [16-20].

In this paper, based on a new multiplication of fractional analytic functions, we obtain the following matrix fractional integral:

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(tA\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha p}\otimes_\alpha E_\alpha\left(\left(tA\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha q}\right)\right],$$

where $0<\alpha\le 1$, p,q are positive integers, t is a real number, and A is a real matrix. Moreover, our result is a generalization of the result in ordinary calculus.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper.

Definition 2.1 ([21])**:** Let $0<\alpha\le 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\,. \tag{1}$$

And the Jumarie type of Riemann-Liouville α-fractional integral is defined by

$$\left({}_{x_0}I_{x}^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}\frac{f(t)}{(x-t)^{1-\alpha}}dt\,, \tag{2}$$

where $\Gamma(\ )$ is the gamma function.

Definition 2.2 ([22])**:** If x, x_0, and a_n are real numbers for all n, $x_0\in(a,b)$, and $0<\alpha\le 1$. If the function $f_\alpha:[a,b]\to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha)=\sum_{n=0}^{\infty}\frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval

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containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.3 ([23])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}, \tag{3}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha}. \tag{4}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)} (x - x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) (x - x_0)^{n\alpha}. \end{aligned} \tag{5}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!} \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{m=0}^{n} \binom{n}{m} a_{n-m} b_m \right) \left(\frac{1}{\Gamma(\alpha+1)} (x - x_0)^\alpha \right)^{\otimes_\alpha n}. \end{aligned} \tag{6}$$

Definition 2.4 ([24])**:** If $0 < \alpha \leq 1$, and A is a matrix. The matrix α-fractional exponential function is defined by

$$E_\alpha(Ax^\alpha) = \sum_{n=0}^{\infty} A^n \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(A \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha n}. \tag{7}$$

III. MAIN RESULT

In this section, we find some type of matrix fractional integral.

Theorem 3.1: *Let* $0 < \alpha \leq 1$, p, q *be positive integers*, t *be a real number, and* A *be a matrix. Then*

$$\left({}_0I_x^\alpha \right) \left[\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha p} \otimes_\alpha E_\alpha \left(\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha q} \right) \right] = \sum_{n=0}^{\infty} \frac{1}{n!(nq+p+1)} (tA)^{nq+p} \left(\frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (nq+p+1)}. \tag{8}$$

Proof

$$\begin{aligned} & \left({}_0I_x^\alpha \right) \left[\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha p} \otimes_\alpha E_\alpha \left(\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha q} \right) \right] \\ &= \left({}_0I_x^\alpha \right) \left[\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha p} \otimes_\alpha \sum_{n=0}^{\infty} \frac{1}{n!} \left(\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha q} \right)^{\otimes_\alpha n} \right] \\ &= \left({}_0I_x^\alpha \right) \left[\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha p} \otimes_\alpha \sum_{n=0}^{\infty} \frac{1}{n!} \left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha nq} \right] \\ &= \left({}_0I_x^\alpha \right) \left[\sum_{n=0}^{\infty} \frac{1}{n!} \left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (nq+p)} \right] \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left({}_0I_x^\alpha \right) \left[\left(tA \frac{1}{\Gamma(\alpha+1)} x^\alpha \right)^{\otimes_\alpha (nq+p)} \right] \end{aligned}$$

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$$= \sum_{n=0}^{\infty} \frac{1}{n!} (tA)^{nq+p} \left({}_{0}I_{x}^{\alpha} \right) \left[\left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (nq+p)} \right]$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!(nq+p+1)} (tA)^{nq+p} \left(\frac{1}{\Gamma(\alpha+1)} x^{\alpha} \right)^{\otimes_\alpha (nq+p+1)} .$$ q.e.d.

IV. CONCLUSION

In this paper, based on a new multiplication of fractional analytic functions, we evaluate some type of matrix fractional integral. In fact, our result is a generalization of classical calculus result. In the future, we will continue to use the new multiplication of fractional analytic functions to solve the problems in applied mathematics and fractional differential equations.

REFERENCES

[1] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, vol. 5, no. 2, pp, 41-45, 2016.

[2] R. L. Magin, Fractional calculus models of complex dynamics in biological tissues, Computers & Mathematics with Applications, vol. 59, no. 5, pp. 1586-1593, 2010.

[3] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[4] F. C. Meral, T. J. Royston, R. Magin, Fractional calculus in viscoelasticity: an experimental study, Communications in Nonlinear Science and Numerical Simulation, vol. 15, no. 4, pp. 939-945, 2010.

[5] R. Hilfer (ed.), Applications of Fractional Calculus in Physics, WSPC, Singapore, 2000.

[6] V. V. Uchaikin, Fractional Derivatives for Physicists and Engineers, Vol. 1, Background and Theory, Vol. 2, Application. Springer, 2013.

[7] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, Vol. 8, No. 5, 660, 2020.

[8] M. F. Silva, J. A. T. Machado, A. M. Lopes, Fractional order control of a hexapod robot, Nonlinear Dynamics, vol. 38, pp. 417-433, 2004.

[9] A. Carpinteri, F. Mainardi, (Eds.), Fractals and fractional calculus in continuum mechanics, Springer, Wien, 1997.

[10] N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, Journal of Vibration and Control, vol. 14, no. 9, pp. 1587-1596, 2008.

[11] R. C. Koeller, Applications of fractional calculus to the theory of viscoelasticity, Journal of Applied Mechanics, vol. 51, no. 2, 299, 1984.

[12] T. Sandev, R. Metzler, & Ž. Tomovski, Fractional diffusion equation with a generalized Riemann–Liouville time fractional derivative, Journal of Physics A: Mathematical and Theoretical, vol. 44, no. 25, 255203, 2011.

[13] J. P. Yan, C. P. Li, On chaos synchronization of fractional differential equations, Chaos, Solitons & Fractals, vol. 32, pp. 725-735, 2007.

[14] C. -H. Yu, A study on fractional RLC circuit, International Research Journal of Engineering and Technology, vol. 7, no. 8, pp. 3422-3425, 2020.

[15] C. -H. Yu, A new insight into fractional logistic equation, International Journal of Engineering Research and Reviews, vol. 9, no. 2, pp.13-17, 2021.

[16] K. S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations; John Willy and Sons, Inc.: New York, NY, USA, 1993.

[17] K. B. Oldham, J. Spanier, The Fractional Calculus; Academic Press: New York, NY, USA, 1974.

[18] I. Podlubny, Fractional Differential Equations; Academic Press: New York, NY, USA, 1999.

[19] S. Das, Functional Fractional Calculus, 2nd Edition, Springer-Verlag, 2011.

[20] K. Diethelm, The Analysis of Fractional Differential Equations, Springer-Verlag, 2010.

[21] C. -H, Yu, Study of fractional Fourier series expansions of two types of matrix fractional functions, International Journal of Mathematics and Physical Sciences Research, vol. 12, no. 2, pp. 13-17, 2024.

[22] C. -H, Yu, Fractional partial differential problem of some matrix two-variables fractional functions, International Journal of Mechanical and Industrial Technology, vol. 12, no. 2, pp. 6-13, 2024.

[23] C. -H, Yu, Study of two matrix fractional integrals by using differentiation under fractional integral sign, International Journal of Civil and Structural Engineering Research, vol. 12, no. 2, pp. 24-30, 2024.

[24] C. -H, Yu, Fractional differential problem of two matrix fractional hyperbolic functions, International Journal of Recent Research in Civil and Mechanical Engineering, vol. 11, no. 2, pp. 1-4, 2024.